Sankhyā : The Indian Journal of Statistics
1980, Volume 42, Series B, Pts. 3 and 4, pp. 197–216.

ON A CLASS OF MULTISTAGE SELECTION PROCEDURES WITH SCREENING FOR THE NORMAL MEANS PROBLEM*

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SUMMARY. The problem of selecting the population associated with the largest mean from k normal populations which have a common known variance is considered. A class of L-stage selection procedures, which have the desirable property that they screen out the non-contending populations as the sampling proceeds from one stage to the next, is proposed. The proposed procedures are adaptive, capitalizing on favorable configurations of population means, and have the added advantage that they terminate in at most L stages (where L is typically small, two to five). Tables of "optimal" design constants required to implement the procedures are provided as are the performance assessments based on Monte Carlo simulations for the procedures using these design constants. The proposed procedures are compared with some existing procedures and it is found that the former offer considerable improvement over the latter in large number of situations.

1. Introduction

In this paper we propose a class of L-stage procedures ($\mathscr{P}_L$) for selecting the normal population with the largest mean (referred to as the "best" population) when the populations have a common known variance. The procedure $\mathscr{P}_L$ screens out the noncontending populations as the sampling proceeds from one stage to the next and employs Gupta's (1956, 1965) screening type subset selection approach in the first (L–1) stages and Bechhofer's (1954) indifference-zone approach to all populations retained in the final stage. This procedure is a generalization of the two-stage procedure $\mathscr{P}_2$ proposed in Bechhofer and Tamhane (1977) (see also Alam, 1970). The practical considerations which prompted the development of multistage screening type selection procedures are explained in Bechhofer and Tamhane (*op. cit.*).

This same approach is applicable to many other selection problems, e.g. normal variances problem. Also by using appropriate transformations to normality, the tables given in this paper can be applied as approximations in these other selection problems. Now we give the mathematical formulation of the selection problem and some notation.

*Research supported by NSF Grant No. ENG 77-061 12.

Let X_{iv} ($v = 1, 2, \ldots$) denote i.i.d observations from population Π_i which is normal with unknown mean μ_i ($1 \leqslant i \leqslant k$) and known variance σ^2. Let $\mu_{[1]} \leqslant \ldots \leqslant \mu_{[k]}$ denote the ordered values of the μ_i and let $\delta_{ij} = \mu_{[i]} - \mu_{[j]}$ ($1 \leqslant i, j \leqslant k$). It is not known which population is associated with $\mu_{[i]}$ ($1 \leqslant i \leqslant k$). Experimenter's *goal* is to select the population associated with $\mu_{[k]}$ (if two or more populations tie for the largest mean then select any one of those populations), termed the *best* population. Such a selection is referred to as a *correct selection* (CS).

According to the *indifference-zone approach* (Bechhofer, 1954) to this selection problem, it is assumed that the experimenter can preassign two constants $\delta^* > 0$ and P^* ($1/k < P^* < 1$). He wants a procedure $\mathscr{P}$ which guarantees the *probability requirement*

$$P_{\boldsymbol{\mu}}(\mathrm{CS} \mid \mathscr{P}) \geqslant P^* \text{ whenever } \delta_{k,\,k-1} \geqslant \delta^*, \qquad \ldots \quad (1.1)$$

where the subscript $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ denotes that the probability is computed under the parameter configuration $\boldsymbol{\mu}$.

The *parameter space* of the possible $\boldsymbol{\mu}$'s is denoted by Ω and the *preference zone* for a CS by $\Omega(\delta^*) = \{\boldsymbol{\mu} \in \Omega \mid \delta_{k,\,k-1} \geqslant \delta^*\}$. The associated *indifference zone* is the complement of $\Omega(\delta^*)$ in Ω. To solve the mathematical problem of guaranteeing (1.1) for a specified $\mathscr{P}$ it is necessary to find a $\boldsymbol{\mu}_0 \in \Omega(\delta^*)$ where the infimum of $P_{\boldsymbol{\mu}}(\mathrm{CS} \mid \mathscr{P})$ over $\Omega(\delta^*)$ occurs; such a $\boldsymbol{\mu}_0$ is referred as a *least favourable configuration* (LFC) for $\mathscr{P}$.

A brief summary of the paper is as follows : In Section 2 we propose a L-stage procedure $\mathscr{P}_L$ and derive an expression for its exact PCS when $k = 2$ and lower bounds on the PCS for $k > 2$. Section 3 gives the various restrictions imposed on the design constants of $\mathscr{P}_L$ to facilitate their evaluation. Section 4 describes a method for determining the "optimal" design constants of $\mathscr{P}_L$ by minimizing the expected total sample size at some selected $\boldsymbol{\mu}$; the estimates of the expected total sample size are obtained by Monte Carlo simulations. Finally Section 5 gives some performance comparisons.

2. A L-STAGE PROCEDURE $\mathscr{P}_L$ AND ITS *PCS*

2.1. *A L-stage procedure* $\mathscr{P}_L$. We propose a L-stage ($L \geqslant 2$) procedure $\mathscr{P}_L = \mathscr{P}_L(N_1, \ldots, N_L;\ h_1, \ldots, h_L)$ which depends on positive integers $N_1 \leqslant \ldots \leqslant N_L$ and real constants $h_l \geqslant 0$ ($1 \leqslant l \leqslant L$) with $h_L \equiv 0$; these are referred to as the design constants of $\mathscr{P}_L$. The integer $n_l = N_l - N_{l-1}$

(where $N_0 \equiv 0$) denotes the common number of observations to be taken at the l-th stage ($1 \leqslant l \leqslant L$) from each population still retained for sampling at that stage and h_l is the "yardstick" used in determining the subset of the populations to be retained in the $(l+1)$st stage ($1 \leqslant l \leqslant L-1$), the sampling being terminated at the L-th stage. The design constants of $\mathscr{P}_L$ are to be determined so that (1.1) is guaranteed. Now we describe $\mathscr{P}_L$.

At the 1st stage take a random sample of size $n_1 = N_1$ from each Π_i for $i \in I_1 = \{1, 2, \ldots, k\}$ and compute the k sample means $\bar{X}_i^{(1)} = \sum_{\nu=1}^{N_1} X_{i\nu}/N_1$ ($1 \leqslant i \leqslant k$). Determine the subset $\{\Pi_i,\ i \in I_2\}$ to be retained for sampling in the second stage where $I_2 = \{i \in I_1 \mid \bar{X}_i^{(1)} \geqslant \max_{j \in I_1} \bar{X}_j^{(1)} - h_1\}$. If only a single population is retained then stop sampling and assert that, that population is best. Otherwise proceed to the second stage. In general, at the l-th stage ($1 \leqslant l \leqslant L-1$), take a random sample of size $n_l = N_l - N_{l-1}$ from each Π_i for $i \in I_l$ (i.e., the set of populations retained for sampling at the l-th stage) and compute the *cumulative* sample means $\bar{X}_i^{(l)} = \sum_{\nu=1}^{N_l} X_{i\nu}/N_l$. Determine the subset $\{\Pi_i,\ i \in I_{l+1}\}$ of populations to be retained for sampling in the $(l+1)$st stage where $I_{l+1} = \{i \in I_l \mid \bar{X}_i^{(l)} \geqslant \max_{j \in I_l} \bar{X}_j^{(l)} - h_l\}$. If only a single population is retained then stop sampling and assert that, that population is best. Otherwise proceed to the $(l+1)$st stage. At the L-th stage, take a random sample of size $n_L = N_L - N_{L-1}$ from each Π_i for $i \in I_{L-1}$ and compute the cumulative sample means $\bar{X}_i^{(L)} = \sum_{\nu=1}^{N_L} X_{i\nu}/N_L$. Terminate sampling and assert that the population associated with $\max_{j \in I_L} \bar{X}_j^{(L)}$ is best.

Note that only the case $L > 2$ is of concern to us here because for $L = 2$ the minimax design constants (without placing any restrictions on them as we do for $L > 2$ in the present paper) are given in Tamhane and Bechhofer (1979).

2.2. *Exact PCS for $k = 2$ and lower bounds on the PCS for $k > 2$.* An exact general expression for the PCS of $\mathscr{P}_L$ for arbitrary k and L is difficult to derive. For $L = 2$ and arbitrary k an exact expression for the PCS has been derived in Tamhane and Bechhofer (1977). However, the LFC associated with this exact PCS has been determined only for $k = 2$. Even if the LFC can be determined for $k > 2$, the problem of numerical evaluation of the exact PCS on a computer appears to be very formidable.

For $k = 2$ and arbitrary L, an exact expression for the PCS of $\mathscr{P}_L$ can be derived as follows: Let PCS_l denote the probability that a correct selection is made at the l-th stage ($1 \leqslant l \leqslant L$). Let $\bar{X}_{(i)}^{(l)}$ denote the cumulative sample mean up to the l-th stage which is associated with $\mu_{[i]}$ and let $\tilde{X}_{(i)}^{(l)}$ be the corresponding sample mean based on the observations taken in the l-th stage only ($1 \leqslant l \leqslant L$; $i = 1, 2$). Further let $\delta = \mu_{[2]} - \mu_{[1]}$ and $p_{ij} = n_i/N_j$ ($1 \leqslant i \leqslant j$, $1 \leqslant j \leqslant L$). Then

$$P_{\boldsymbol{\mu}}(\mathrm{CS} \mid \mathscr{P}_L) = \sum_{l=1}^{L} \mathrm{PCS}_l \qquad \ldots \quad (2.1)$$

where

$$\mathrm{PCS}_1 = \Phi\left[\frac{(\delta - h_1)}{\sigma}\sqrt{\frac{N_1}{2}}\right] \qquad \ldots \quad (2.2)$$

and

$$\mathrm{PCS}_l = P_{\boldsymbol{\mu}}\{\bar{X}_{(2)}^{(l)} > \bar{X}_{(1)}^{(l)} + h_l\,;\ -h_j \leqslant \bar{X}_{(1)}^{(j)} - \bar{X}_{(2)}^{(j)} \leqslant h_j\ (1 \leqslant j \leqslant l-1)\}$$

$$= P_{\boldsymbol{\mu}}\left\{\sum_{j=1}^{l}\sqrt{p_{jl}}\left[\frac{(\tilde{X}_{(1)}^{(j)} - \tilde{X}_{(2)}^{(j)} + \delta)}{\sigma}\sqrt{\frac{n_j}{2}}\right] < \frac{(\delta - h_l)}{\sigma}\sqrt{\frac{N_l}{2}}\,;\ \frac{(\delta - h_j)}{\sigma}\sqrt{\frac{N_j}{2}}\right.$$

$$\left. < \sum_{i=1}^{j}\sqrt{p_{ij}}\left[\frac{(\tilde{X}_{(1)}^{(i)} - \tilde{X}_{(2)}^{(i)} + \delta)}{\sigma}\sqrt{\frac{n_i}{2}}\right] < \frac{(\delta + h_j)}{\sigma}\sqrt{\frac{N_j}{2}}\ (1 \leqslant j \leqslant l-1)\right\}$$

$$= \int \cdots \int \Phi\left[\frac{1}{\sqrt{p_{ll}}}\left\{\frac{(\delta - h_l)}{\sigma}\sqrt{\frac{N_l}{2}} - \sum_{j=1}^{l-1}\sqrt{p_{jl}}\,x_j\right\}\right]\prod_{j=1}^{l-1}\phi(x_j)dx_j$$

$$\ldots \quad (2.3)$$

for $2 \leqslant l \leqslant L$. In (2.3) $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal c.d.f and the corresponding p.d.f respectively and the lower and upper limits for the j-th ($1 \leqslant j \leqslant l$-1) integral are

$$\frac{1}{\sqrt{p_{jj}}}\left\{\frac{(\delta - h_j)}{\sigma}\sqrt{\frac{N_j}{2}} - \sum_{i=1}^{j-1}\sqrt{p_{ij}}x_i\right\}$$

and

$$\frac{1}{\sqrt{p_{jj}}}\left\{\frac{(\delta + h_j)}{\sigma}\sqrt{\frac{N_j}{2}} - \sum_{i=1}^{j-1}\sqrt{p_{ij}}x_i\right\}$$

respectively.

By writing

$$P_{\mu}(\mathrm{CS}\,|\,\mathscr{P}_L) = P_{\mu}\{\Pi_{(1)} \text{ gets eliminated}\}$$

$$= P_{\mu}\{\bar{X}^{(l)}_{(1)} - \bar{X}^{(l)}_{(2)} < -h_l \text{ for some } l\,(1 \leqslant l \leqslant L)\}$$

where $\Pi_{(1)}$ is the population with mean $\mu_{[1]}$, it is easy to see that $P_{\mu}(\mathrm{CS}\,|\,\mathscr{P}_L)$ is an increasing function of $\delta = \mu_{[2]} - \mu_{[1]}$ for $k = 2$. Therefore the LFC is given by $\delta = \delta^*$ for $k = 2$.

The case $k = 2$ is not of much practical interest (because there is no screening involved), but we can use the results for this case to give us some idea about how much we lose by using a lower bound on the PCS to be proposed in the theorem below for arbitrary k and L. For both the bounds proposed in the theorem, namely (2.4) and (2.5), it can be shown that the LFC is the usual slippage configuration; see the corollary. However, only the infimum of the less sharp bound given by (2.5) is computable in practice. We have given the sharper bound here with the hope that computations with it will become feasible in future. The proof of the theorem is omitted because it is a straightforward extension of Theorem 3.1 in Tamhane and Bechhofer (1979).

Theorem : *For any* $\mu \in \Omega$ *we have the following inequalities* :

$$P_{\mu}(\mathrm{CS}\,|\,\mathscr{P}_L) \geqslant \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi_L \left[\frac{(\delta_{kl} + h_1)\sqrt{N_1}}{\sigma} - x_1, \ldots, \right.$$

$$\left. \frac{(\delta_{kl} + h_L)\sqrt{N_L}}{\sigma} - x_L \,|\, \boldsymbol{R}_L \right] \times \phi_L(x_1, \ldots, x_L \,|\, \boldsymbol{R}_L) dx_1 \ldots dx_L \quad \ldots \quad (2.4)$$

$$\geqslant \prod_{l=1}^{L} \left\{ \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi \left[x + \frac{(\delta_{kl} + h_l)\sqrt{N_i}}{\sigma} \right] d\Phi(x) \right\} \quad \ldots \quad (2.5)$$

where $\Phi_L(\cdot, \ldots, \cdot \,|\, \boldsymbol{R}_L)$ *denotes the c.d.f. and* $\phi_L(\cdot, \ldots, \cdot \,|\, \boldsymbol{R}_L)$ *the corresponding p.d.f. of a L-variate random vector whose components are standard normal and whose correlation matrix is* $\boldsymbol{R}_L = \{r_{lm}\}$ *where*

$$r_{lm} = r_{ml} = \sqrt{N_l / N_m} \quad (1 \leqslant l < m \leqslant L). \quad \ldots \quad (2.6)$$

Corollary : *The infimum with respect to* μ *of* (2.4) *and* (2.5) *over* $\Omega(\delta^*)$ *is attained at the configuration* $\mu_{[1]} = \mu_{[k-1]} = \mu_{[k]} - \delta^*$ *(LFC) and thus for all* $\mu \in \Omega(\delta^*)$ *we have that*

$$P_\mu(\mathrm{CS} \mid \mathscr{P}_L) \geqslant \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_L^{k-1} \left[\frac{(\delta^* + h_1)\sqrt{N_1}}{\sigma} - x_1, \ldots, \right.$$

$$\left. \frac{(\delta^* + h_L)\sqrt{N_L}}{\sigma} - x_L \mid \boldsymbol{R}_L \right] \times \phi_L(x_1, \ldots, x_L \mid \boldsymbol{R}_L) dx_1 \ldots dx_L \qquad \ldots \quad (2.7)$$

$$\geqslant \prod_{l=1}^{L} \left\{ \int_{-\infty}^{\infty} \Phi^{k-1} \left[x + \frac{(\delta^* + h_l)\sqrt{N_l}}{\sigma} \right] d\Phi(x) \right\}. \qquad \ldots \quad (2.8)$$

Henceforth we shall use only the bound (2.8) and derive a conservative procedure by equating it to P^*. Note that the discrepancy between (2.7) and (2.8) will increase as L increases but for values of L of concern in this paper (namely three to five) the discrepancy is not substantial.

2.3. *Comparison of the exact PCS and the lower bound for* $k = 2$. For reasons mentioned in Section 2.2 we shall compute the values of the exact PCS for $k = 2$ given by (2.1) (at $\delta = \delta^*$) and the lower bound on it given by (2.8) for some selected values of design constants. Although the exact PCS can not be computed directly for $k > 2$, it can be estimated by Monte Carlo simulations and these estimates can be used to assess the conservativism of the bound; this is done in Section 5.

For the numerical comparison between the exact PCS and the lower bound the design constants of $\mathscr{P}_L$ were selected as follows : We set

$$\frac{(\delta^* + h_1)\sqrt{N_1}}{\sigma} = \ldots = \frac{(\delta^* + h_L)\sqrt{N_L}}{\sigma} = b \text{ (say)},$$

(the above choice is employed later in Section 3) and $n_1 = \ldots = n_L = n$ (say) which need not be an integer. Equating (2.8) to P^* we find that

$$b = \sqrt{2}\,\Phi^{-1}[(P^*)^{1/L}].$$

Thus we obtain

$$\frac{(\delta^* + h_l)}{\sigma}\sqrt{\frac{N_l}{2}} = \Phi^{-1}[(P^*)^{1/L}] \quad (1 \leqslant l \leqslant L) \qquad \ldots \quad (2.9)$$

and

$$\frac{(\delta^*-h_l)}{\sigma}\sqrt{\frac{N_l}{2}} = \Phi^{-1}[(P^*)^{1/L}]\left\{2\sqrt{\frac{l}{L}}-1\right\} \quad (1 \leqslant l \leqslant L). \quad \ldots \quad (2.10)$$

Using (2.9) and (2.10) the values of the exact PCS (given by (2.1)) were computed for $L = 3$ and 5 and for selected values of P^*; the results are displayed in Table 1. Note that $L = 3$ requires evaluation of double integrals while $L = 5$ requires evaluation of quadruple integrals over general regions. These integrations were performed using the NIMRAS routine involving repeated applications of Simpson's rule developed by the Academic Computing Center at the University of Wisconsin, Madison.

TABLE 1. COMPARISON OF THE EXACT PCS AND P^* GUARANTEED BY THE LOWER BOUND (2.8) FOR $k = 2$

P^*	exact PCS at $\mu_{[2]} - \mu_{[1]} = \delta^*$	
	$L=3$	$L=5$
0.75	0.8534	0.8703
0.80	0.8802	0.8948
0.85	0.9076	0.9180
0.90	0.9361	0.9470
0.95	0.9661	0.9705
0.99	0.9924	0.9939

These computations show that the bound is quite conservative for low values of P^* but for $P^* \geqslant 0{\cdot}90$ (which are the values often employed in practice) the bound is not very conservative. As L increases from three to five, there is a slight increase in the conservativism of the bound as is to be expected. The Monte Carlo simulations for $\mathscr{P}_L$ reported in Section 4 and 5 are carried out for $P^* = 0{\cdot}90$ and $0{\cdot}99$ and thus their results should be credible.

3. Some restrictions on the choice of design constants of $\mathscr{P}_L$

Let us reparamoterize the original design constants to new ones as follows :

$$c_l = \frac{\delta^*\sqrt{N_l}}{\sigma}, \quad d_l = \frac{h_l\sqrt{N_l}}{\sigma} \quad (1 \leqslant l \leqslant L).$$

We shall regard $\{(c_l, d_l),\ 1 \leqslant l \leqslant L\}$ as nonnegative continuous variables for convenience. For determination of $\{(c_l, d_l),\ 1 \leqslant l \leqslant L\}$ one can use an

appropriate criterion, viz., minimization of the expected total sample size at the LFC subject to the constraint (obtained from (2.8)) :

$$\prod_{l=1}^{L} \left\{ \int_{-\infty}^{\infty} \Phi^{k-1}(x+c_l+d_l)d\Phi(x) \right\} \geqslant P^*. \qquad \ldots \quad (3.1)$$

The resulting optimization problem will, however, be quite intractable because of the number of variables involved.

To simplify the optimization problem we shall reduce its dimensionality by making the choice

$$c_1+d_1 = c_2+d_2 = \ldots = c_L+d_L = c_L = b \text{ (say)} \qquad \ldots \quad (3.2)$$

where, for given k and P^*, b solves the equation

$$\int_{-\infty}^{\infty} \Phi^{k-1}(x+b)d\Phi(x) = (P^*)^{1/L}. \qquad \ldots \quad (3.3)$$

We shall perform optimization within the restricted class of $\mathcal{P}_L$ defined by (3.2). Indeed, this leads to a suboptimal solution, but, as is found later in the Monte Carlo study, even this solution provides a significant improvement and it has the obvious advantage of less computation and tabulation.

For $k = 3, 4, 5, 7, 10\ (5)\ 25$, $L = 3, 4, 5$ and $P^* = 0{\cdot}75, 0{\cdot}90, 0{\cdot}95$ and $0{\cdot}99$, the values of b calculated from (3.3) are given in Table 2. One can also use tables in Gupta (1963) or Milton (1963) to obtain the b-values in an approximate fashion. The values of (c_l, d_l) for $1 \leqslant l \leqslant L$ are given by

$$c_L = b.\ c_l = r_{lL}\ c_L,\ d_l = (1-r_{lL})b \qquad \ldots \quad (3.4)$$

and the values of (N_l, h_l) for $1 \leqslant l \leqslant L$ are given by

$$N_L = [(b\sigma/\delta^*)^2],\ N_l = [(r_{lL}\, b\sigma/\delta^*)^2],\ h_l = \delta^*/(1-r_{lL})/r_{lL} \qquad \ldots \quad (3.5)$$

where $[x]$ denotes the smallest integer $\geqslant x$.

The experimenter can use tabulated values of b in conjunction with *any* choice of sample size ratios to obtain (N_l, h_l) from (3.5) which would guarantee (1.1). We consider a special family of sample size ratios, namely that the sample sizes in successive stages are in constant proportion, i.e. $n_l = a^{l-1}n_1$ $(1 \leqslant l \leqslant L)$ for some $a > 0$. Thus $r_{lL} = \{(1-a^l)/(1-a^L)\}^{\frac{1}{2}}$ for $a \neq 1$ and $r_{lL} = \sqrt{l/L}$ for $a = 1$.

TABLE 2. DESIGN CONSTANTS b

k	L	P^*			
		0.75	0.90	0.95	0.99
3	3	2.2966	2.9401	3.3432	4.1346
	4	2.4924	3.1059	3.4936	4.2608
	5	2.6375	3.2302	3.6070	4.3568
4	3	2.5158	3.1392	3.5305	4.3007
	4	2.7054	3.3001	3.6767	4.4237
	5	2.8458	3.4208	3.7869	4.5173
5	3	2.6627	3.2739	3.6578	4.4146
	4	2.8484	3.4317	3.8014	4.5357
	5	2.9862	3.5502	3.9097	4.6278
7	3	2.8586	3.4551	3.8303	4.5704
	4	3.0399	3.6093	3.9706	4.6889
	5	3.1743	3.7250	4.0764	4.7791
10	3	3.0430	3.6273	3.9949	4.7205
	4	3.2206	3.7784	4.1324	4.8368
	5	3.3522	3.8917	4.2361	4.9253
15	3	3.2324	3.8055	4.1662	4.8779
	4	3.4065	3.9537	4.3010	4.9923
	5	3.5357	4.0649	4.4028	5.0791
20	3	3.3569	3.9235	4.2797	4.9836
	4	3.5291	4.0700	4.4132	5.0965
	5	3.6567	4.1799	4.5138	5.1824
25	3	3.4490	4.0111	4.3646	5.0623
	4	3.6198	4.1563	4.4968	5.1745
	5	3.7464	4.2654	4.5966	5.2597

This special choice of sample size ratios appears to be reasonable; furthermore, it enables us to seek for an "optimal" (with respect to an appropriate criterion) choice of these ratios by manipulating a single quantity a. It might be noted that within this further restricted class, for given k and P^*, $\mathcal{P}_L$ depends only on a; we denote the corresponding $\mathcal{P}_L$ by $\mathcal{P}_L(a)$.

4. Determination of optimal constants a^*

4.1. *Expected number of stages and expected sample size for $\mathcal{P}_L$.* Let M and N denote, respectively, the number of stages to termination and the total sample size required by $\mathcal{P}_L$. Let $E_{\mu}(M \mid \mathcal{P}_L)$ and $E_{\mu}(N \mid \mathcal{P}_L)$ denote the corresponding expected values evaluated at μ. These two quantities are later used in assessing the performance of $\mathcal{P}_L$.

Although it is possible to derive exact expressions for $E_{\mu}(M \mid \mathscr{P}_L)$ and $E_{\mu}(N \mid \mathscr{P}_L)$, they are too cumbersome to be of much computational or analytical use. Therefore we have used the Monte Carlo estimates of these quantities in the performance assessment of $\mathscr{P}_L$. Exact expressions of these quantities are derived in Tamhane (1978) for some special cases ($L = 2$, $k \geqslant 2$ and $L > 2$, $k = 2$).

4.2. *Monte Carlo results and a table of optimal constants a^*.* To determine the "optimal" values of a, we have considered two criteria : (i) minimize $E(N)$ at the configuration $\mu_{[1]} = \mu_{[k]}$ called the *equal means configuration* (EMC); (ii) minimize $E(N)$ at the configuration $\mu_{[1]} = \mu_{[k-1]} = \mu_{[k]} - \delta^*$ called the Although $E(M)$ is not considered in these optimization criteria, it is taken into account later in the overall performance assessment of $\mathscr{P}_L$ made in Section 5.

The optimal values $a^*(a^*_{EMC}$ minimizes $E_{EMC}(N \mid \mathscr{P}_L)$ and a^*_{LFC} minimizes $E_{LFC}(N \mid \mathscr{P}_L)$ were found by a search method which used the estimates of $E(N \mid \mathscr{P}_L)$ at the respective configurations obtained from Monte Carlo experiments. For specified values of δ^*, P^*, k and σ^2, the search was conducted by varying the value of a in the appropriate direction in steps of 0·1. Because of budget limitations, we have found the a^*-values only for $k = 5$, 10, 25; $L = 3$, 5 and $P^* = 0{\cdot}90$ and 0·99. These values are given in Table 3. It should be noted that the variation in a^*-values is very small and hence, interpolation can be used for $P^* = 0{\cdot}95$ and the other (k, L)-combinations.

For each (k, L, P^*)-combination T Monte Carlo experiments were conducted where $T = 1000$ or 500 as given in Tables 4 and 5. Throughout δ^* and σ^* were fixed at 1. (The fact that this choice is not restrictive is clear if we note that both $E_{\mu}(M \mid \mathscr{P}_L)$ and $E_{\mu}(N \mid \mathscr{P}_L)$ depend on the distribution of differences between $\bar{X}^{(l)}_i$ and $\bar{X}^{(l)}_j$ and if we consider an appropriate standardization.) The Fortran routine RANF was used to generate the uniform [0, 1] random variables which were used in the Box-Müller algorithm to generate the standard normal random variables. It was observed in all cases that, as a is increased, $E_{EMC}(N \mid \mathscr{P}_L)$ ($E_{LFC}(N \mid \mathscr{P}_L)$) first decreases and after achieving a certain minimum value increases. Thus no multiple minima were observed. In most cases a^*_{EMC} and a^*_{LFC} are not too different with $a^*_{EMC} \leqslant a^*_{LFC}$ in all cases. This latter fact helped in restricting the search. An explanation for this fact is that when the true μ = LFC, fewer populations are retained for sampling in the latter stages of $\mathscr{P}_L$ compared to when the true μ = EMC. Thus $E_{LFC}(N \mid \mathscr{P}_L)$ is made smaller by taking fewer

TABLE 3. DESIGN CONSTANTS A*

k	P^*	L	a^*_{EMC}	a^*_{LFC}
5	0.90	3	1.2	1.5
		5	1.2	1.4
	0.99	3	0.9	1.4
		5	1.0	1.2
10	0.90	3	1.4	1.6
		5	1.1	1.3
	0.99	3	1.0	1.3
		5	1.0	1.3
25	0.90	3	1.5	1.6
		5	1.3	1.3
	0.99	3	1.1	1.4
		5	1.1	1.3

observations on each population in the earlier stages and more observations on each population retained in the latter stages which corresponds to choosing larger value for a^*_{LFC} than for a^*_{EMC}.

For each a^*_{EMC} and a^*_{LFC}, the estimates of $E(M)$ and $E(N)$ were obtained at three configurations: EMC, LFC and a configuration called the *equal distance configuration* or EDC which satisfies $\mu_{[i+1]}-\mu_{[i]} = \delta^*$ $(1 \leqslant i \leqslant k-1)$. In addition the estimates of the achieved PCS at LFC were obtained. All these estimates along with their standard errors are given in Table 4 (for (a^*_{EMC}) and Table 5 (for a^*_{LFC}). A discussion of the simulation results is given in Section 5.2.

5. Some performance comparisons

5.1. *Comparison of $\mathscr{P}_L(a^*_{EMC})$ and $\mathscr{P}_L(a^*_{LFC})$ with $\mathscr{P}_1$.* As a measure of the performance of $\mathscr{P}_L(\mathscr{P}_L(a^*_{EMC})$ or $\mathscr{P}_L(a^*_{LFC})$) relative to that of Bechhofer's (1954) single-stage procedure $\mathscr{P}_1$ when both guarantee the same probability requirement (1.1) we define the relative efficiency (RE) of $\mathscr{P}_1$ with respect to $\mathscr{P}_L$ as follows:

$$\mathrm{RE}_{\mu}(\mathscr{P}_1 : \mathscr{P}_L \mid k, \delta^*, P^*, \sigma, \gamma) = \frac{\gamma E_{\mu}(M \mid \mathscr{P}_L)+E_{\mu}(N \mid \mathscr{P}_L)}{\gamma \cdot 1+kn} \quad \ldots \quad (5.1)$$

where kn is the total sample size required by $\mathscr{P}_1$ to guarantee (1.1);

$$n = [(c\sigma/\delta^*)^2]$$

where c solves the equation

$$\int_{-\infty}^{\infty} \Phi^{k-1}(x+c)d\Phi(x) = P^*; \qquad \ldots \quad (5.2)$$

and $\gamma \geqslant 0$ can be interpreted as the relative cost of sampling one *stage* versus the cost of taking one *observation*. We are assuming here that γ remains fixed regardless of the number of populations sampled in each stage; perhaps a more realistic measure would take into account the number of populations sampled in each stage. Note that RE-values less than unity favor $\mathscr{P}_L$ over $\mathscr{P}_1$.

Based on the Monte Carlo estimates of $E(M)$ and $E(N)$ given in Tables 4 and 5, the estimates of RE of $\mathscr{P}_1$ relative to $\mathscr{P}_L(a^*_{EMC})$ and $\mathscr{P}_L(a^*_{LFC})$ at EMC, LFC and EDC were computed for $\gamma = 0$, 5 and 10. These RE-values are given in Table 6 (for $\mathscr{P}_1$ relative to $\mathscr{P}_L(a^*_{EMC})$) and Table 7 (for $\mathscr{P}_1$ relative to $\mathscr{P}_L(a^*_{LFC})$). In these computations the c-values which are the solutions to (5.2) were taken from Milton (1963). Now we discuss the performance of $\mathscr{P}_L(a^*_{EMC})$ and $\mathscr{P}_L(a^*_{LFC})$ based on the results in Tables 4 and 6 and Tables 5 and 7 respectively.

First looking at the values of the achieved PCS at LFC in Tables 4 and 5 we see that, as expected, both the $\mathscr{P}_L$'s overprotect in terms of the P^*-requirement and the overprotection increases with L. Clearly at $P^* = 0{\cdot}99$ the overprotection is much less than that at $P^* = 0{\cdot}90$. In either case the bound does not appear to be overly conservative at these P^*-values. Note that in Table 5 when $k = 25$, $P^* = 0{\cdot}99$ and $L = 5$ the observed fact that the achieved PCS $= 0{\cdot}988$ is less than $P^* = 0{\cdot}99$ can be explained as being due to sampling error.

Next we study Tables 6 and 7 which provide summaries of the results in Tables 4 and 5, respectively. In discussing the variations in RE with respect to a certain quantity, say P^*, we assume that the remaining quantities, in this case k, L, γ, μ, a^* and δ^*/σ are kept fixed. Unless otherwise noted, the nature of variation in RE with respect to a given quantity is the same regardless of the values at which the other quantities are kept fixed.

We note that RE decreases as k increases; i.e., savings due to using $\mathscr{P}_L$ in place of $\mathscr{P}_1$ increase with k. This indicates that the effectiveness of $\mathscr{P}_L$ as a screening procedure increases with k. We also note that RE decreases as the δ_{ki} increase and thus $\mathscr{P}_L$ capitalizes on favorable configurations. An important observation is that at $\gamma = 0$, RE < 1 at EMC (which is in some

TABLE 4. MONTE CARLO RESULTS FOR $\pi_L(a^*_{EMC})$

k	P*	L	T	$P_{LFC}(CS)$	EMC		LFC		EDC	
					E(N)	E(M)	E(N)	E(M)	E(N)	E(M)
5	0.90	3	1000	.934	31.883	2.469	24.881	1.958	18.846	1.539
				(.008)	(.276)	(.021)	(.245)	(.021)	(.158)	(.020)
		5	1000	.943	33.021	3.964	24.061	3.072	15.373	2.316
				(.007)	(.324)	(.031)	(.282)	(.033)	(.180)	(.032)
	0.99	3	1000	.991	67.097	2.515	45.941	1.572	39.249	1.253
				(.003)	(.430)	(.020)	(.345)	(.018)	(.186)	(.014)
		5	1000	.994	67.172	4.033	41.497	2.409	29.251	1.861
				(.002)	(.470)	(.029)	(.388)	(.026)	(.230)	(.025)
10	0.90	3	1000	.918	72.018	2.767	56.883	2.298	36.292	1.640
				(.009)	(.516)	(.014)	(.500)	(.019)	(.209)	(.021)
		5	500	.938	72.536	4.212	54.428	3.118	31.846	2.102
				(.011)	(.721)	(.037)	(.697)	(.045)	(.270)	(.041)
	0.99	3	1000	.994	142.24	2.753	100.74	1.852	79.179	1.327
				(.002)	(.753)	(.014)	(.706)	(.019)	(.244)	(.016)
		5	500	.994	138.92	4.332	91.193	2.640	57.541	1.876
				(.003)	(1.14)	(.035)	(1.06)	(.037)	(.360)	(.035)
25	0.90	3	1000	.930	195.35	2.924	159.30	2.511	92.765	1.690
				(.008)	(1.17)	(.008)	(1.18)	(.017)	(.255)	(.020)
		5	500	.932	186.89	4.652	144.33	3.792	60.970	2.454
				(.011)	(1.81)	(.026)	(1.73)	(.035)	(.354)	(.042)
	0.99	3	1000	.993	369.60	2.913	271.16	2.066	200.27	1.387
				(.003)	(1.72)	(.009)	(1.75)	(.016)	(.291)	(.017)
		5	500	.990	349.20	4.646	239.58	3.138	124.59	2.004
				(.004)	(2.68)	(.026)	(2.51)	(.033)	(.411)	(.035)

The standard errors of the estimates are given in round brackets.

sense the "worst" configuration) for all the cases studied. Thus in terms of $E(N)$, both $\pi_L(a^*_{EMC})$ and $\pi_L(a^*_{LFC})$ offer savings over π'_1 regardless of μ.

Next considering variation in RE with respect to γ, we notice that RE increases with γ as is to be expected. However, for $\gamma > 0$, when $k = 5$, RE > 1 only at EMC and in few cases at LFC and when $k = 10$, RE > 1 only in few cases at EMC. In all other cases (in particular at EDC when $k = 5$ or 10 and at all μ when $k = 25$) we notice that RE < 1. Thus even

TABLE 5. MONTE CARLO RESULTS FOR $\mathcal{P}_L\,(a^*_{LFC})$

k	P^*	L	T	P_{LFC}(CS)	EMC		LFC		EDC	
					E(N)	E(M)	E(N)	E(M)	E(N)	E(M)
		3	1000	.931	32.331	2.622	24.456	2.145	16.998	1.706
				(.008)	(.304)	(.019)	(.278)	(.020)	(.184)	(.021)
	0.90									
		5	1000	.943	33.302	4.232	23.782	3.405	14.326	2.677
				(.007)	(.348)	(.028)	(.311)	(.032)	(.200)	(.033)
5										
		3	1000	.991	69.362	2.790	44.130	2.013	30.456	1.598
				(.003)	(.500)	(.014)	(.445)	(.018)	(.266)	(.018)
	0.99									
		5	1000	.993	68.014	4.352	40.464	2.914	25.306	2.261
				(.003)	(.530)	(.026)	(.417)	(.027)	(.257)	(.027)
		3	1000	.931	72.437	2.816	56.803	2.410	32.900	1.744
				(.009)	(.553)	(.013)	(.541)	(.019)	(.233)	(.021)
	0.90									
		5	500	.928	72.988	4.450	53.137	3.554	25.979	2.502
				(.012)	(.806)	(.033)	(.761)	(.042)	(.316)	(.046)
10										
		3	1000	.993	144.33	2.867	97.604	2.112	64.160	1.538
				(.003)	(.852)	(.011)	(.808)	(.017)	(.296)	(.018)
	0.99									
		5	500	.994	141.65	4.658	88.112	3.400	40.747	2.462
				(.003)	(1.29)	(.026)	(1.15)	(.036)	(.417)	(.039)
		3	1000	.928	195.52	2.935	158.68	2.556	87.024	1.769
				(.008)	(1.22)	(.008)	(1.21)	(.017)	(.263)	(.020)
	0.90									
		5	500	.932	186.89	4.652	144.33	3.792	60.970	2.454
				(.011)	(1.81)	(.026)	(1.73)	(.035)	(.354)	(.042)
25										
		3	1000	.991	373.21	2.954	267.54	2.304	157.68	1.609
				(.003)	(1.94)	(.007)	(2.03)	(.016)	(.339)	(.018)
	0.99									
		5	500	.988	353.35	4.828	236.81	3.638	99.882	2.420
				(.005)	(2.89)	(.019)	(2.73)	(.031)	(.450)	(.036)

The standard errors of the estimates are given in round brackets.

TABLE 6. RELATIVE EFFICIENCY OF π_1 WRT π_L (a^*_{EMC})

k	P^*	L	γ	RE_{EMC}	RE_{LFC}	RE_{EDC}
			0	0.9435	0.7363	0.5577
		3	5	1.1401	0.8938	0.6842
			10	1.2918	1.0153	0.7818
	0.90					
			0	0.9772	0.7120	0.4549
		5	5	1.3622	1.0162	0.6948
			10	1.6592	1.2509	0.8799
5						
			0	0.8735	0.5981	0.5109
		3	5	0.9738	0.6576	0.5563
			10	1.0626	0.7103	0.5964
	0.99					
			0	0.8745	0.5402	0.3808
		5	5	1.0675	0.6544	0.4713
			10	1.2383	0.7555	0.5513
			0	0.8094	0.6393	0.4079
		3	5	0.9135	0.7275	0.4734
			10	1.0072	0.8069	0.5324
	0.90					
			0	0.8152	0.6117	0.3579
		5	5	0.9959	0.7450	0.4507
			10	1.1584	0.8649	0.5341
10						
			0	0.7892	0.5589	0.4393
		3	5	0.8422	0.5938	0.4632
			10	0.8924	0.6269	0.4859
	0.99					
			0	0.7707	0.5059	0.3192
		5	5	0.8669	0.5635	0.3613
			10	0.9579	0.6181	0.4011
			0	0.6795	0.5541	0.3227
		3	5	0.7179	0.5875	0.3460
			10	0.7549	0.6199	0.3686
	0.90					
			0	0.6501	0.5020	0.2121
		5	5	0.7185	0.5583	0.2504
			10	0.9410	0.6126	0.2874
25						
			0	0.6969	0.5113	0.3776
		3	5	0.7176	0.5258	0.3871
	0.99		10	0.7380	0.5401	0.3963
			0	0.6585	0.4518	0.2349
		5	5	0.6957	0.4769	0.2515
			10	0.7323	0.5015	0.2677

TABLE 7. RELATIVE EFFICIENCY OF $\mathcal{P}_1$ WRT $\mathcal{P}_L(a^*_{LFC})$

k	P^*	L	γ	RE_{EMC}	RE_{LFC}	RE_{EDC}
		3	0	0.9568	0.7237	0.5030
			5	1.1714	0.9069	0.6581
			10	1.3370	1.0483	0.7777
	0.90					
			0	0.9855	0.7038	0.4239
		5	5	1.4039	1.0519	0.7143
			10	1.7268	1.3206	0.9384
5			0	0.9030	0.5745	0.3965
		3	5	1.0183	0.6624	0.4699
			10	1.1203	0.7402	0.5349
	0.99					
			0	0.8854	0.5268	0.3294
		5	5	1.0973	0.6727	0.4475
			10	1.2847	0.8017	0.5519
			0	0.8141	0.6384	0.3698
		3	5	0.9206	0.7326	0.4429
			10	1.0164	0.8174	0.5086
	0.90					
			0	0.8203	0.5972	0.2920
		5	5	1.0134	0.7545	0.4095
			10	1.1870	0.8959	0.5152
10						
			0	0.8007	0.5415	0.3560
		3	5	0.8565	0.5839	0.3879
			10	0.9093	0.6240	0.4181
	0.99					
			0	0.7859	0.4888	0.2261
		5	5	0.8904	0.5674	0.2864
			10	0.9894	0.6419	0.3436
			0	0.6801	0.5519	0.3027
		3	5	0.7186	0.5862	0.3278
			10	0.7559	0.6193	0.3520
	0.90					
			0	0.6501	0.5020	0.2121
		5	5	0.7185	0.5583	0.2504
			10	0.7846	0.6126	0.2874
25						
			0	0.7037	0.5045	0.2973
		3	5	0.7248	0.5213	0.3096
	0.99		10	0.7454	0.5378	0.3216
			0	0.6663	0.4465	0.1883
		5	5	0.7052	0.4764	0.2092
			10	0.7433	0.5056	0.2296

TABLE 8. RELATIVE EFFICIENCY OF $\mathscr{P}_2^*$ WRT $\mathscr{P}_L$ (a_{EDC}^*)

k	P^*	L	γ	RE_{EMC}	RE_{LFC}	RE_{EDC}
			0	1.0811	0.9824	0.9240
		3	5	1.1319	1.0234	0.9656
			10	1.1627	1.0479	0.9990
	0.90					
			0	1.1197	0.9500	0.7537
		5	5	1.3523	1.1636	0.9806
			10	1.4932	1.2912	1.1145
5						
			0	1.0175	0.9827	0.8220
		3	5	1.0595	0.9409	0.8548
			10	1.0924	0.9714	0.8814
	0.99					
			0	1.0186	0.8155	0.6127
		5	5	1.1614	0.9364	0.7245
			10	1.2731	1.0333	0.8155
			0	1.0166	0.9448	0.7790
		3	5	1.0664	0.9931	0.8379
			10	1.1056	1.0307	0.8839
	0.90					
			0	1.0239	0.9041	0.6836
		5	5	1.1627	1.0171	0.7977
			10	1.2716	1.1049	0.8869
10						
			0	0.9774	0.8927	0.7624
		3	5	1.0054	0.9192	0.7850
			10	1.0301	0.9429	0.8055
	0.99					
			0	0.9546	0.8081	0.5540
		5	5	1.0349	0.8724	0.6121
			10	1.1057	0.9296	0.6648
			0	0.9674	0.9132	0.6924
		3	5	0.9915	0.9365	0.7217
			10	1.0135	0.9577	0.7483
	0.90					
			0	0.9255	0.8273	0.4551
		5	5	0.9921	0.8898	0.5222
			10	1.0533	0.9464	0.5835
25						
			0	0.9477	0.8760	0.7120
		3	5	0.9607	0.8875	0.7226
			10	0.9731	0.8986	0.7329
	0.99					
			0	0.8954	0.7740	0.4429
		5	5	0.9314	0.8049	0.4694
			10	0.9656	0.8344	0.4950

after accounting for the cost of sampling additional stages, $\mathscr{P}_L(a^*_{EMC})$ and $\mathscr{P}_L(a^*_{LFC})$ dominate $\mathscr{P}_1$ in large number of situations. The effect of increasing γ on RE is much less for large values of k. This is because the expected total sample sizes are large for large k and hence the cost of sampling stages is only a small fraction of the total sampling cost.

Next considering variation in RE with respect to L, we notice that, for $\gamma = 0$, RE decreases as L increases except in few cases at EMC when $k = 5$ or 10. Thus increasing the number of stages L results in savings in $E(N)$ at least for moderately large k and at μ-values away from EMC. Clearly for $\gamma > 0$, RE increases with L in many instances because the cost of sampling additional stages is taken into account.

It is also found that, in general RE decreases as P^* increases although there are exceptions to this when $k = 25$. Finally we note that RE relative to $\mathscr{P}_L(a^*_{EMC}) \leqslant$ RE relative to $\mathscr{P}_L(a^*_{LFC})$ for all the three values of γ at EMC (for $\gamma = 0$ by design) and for $\gamma = 5$ and 10 at LFC. On the other hand, RE relative $\mathscr{P}_L(a^*_{LFC}) \leqslant$ RE relative to $\mathscr{P}_L(a^*_{EMC})$ at LFC for $\gamma = 0$ (by design) and at all the three values of γ at EDC (except in few cases when $k = 5$). Thus, in general $\mathscr{P}_L(a^*_{LFC})$ performs better than $\mathscr{P}_L(a^*_{EMC})$ when the μ_i are spread far apart; otherwise $\mathscr{P}_L(a^*_{EMC})$ performs better.

5.2. *Comparison of $\mathscr{P}_L(a^*_{EMC})$ with minimax $\mathscr{P}_2$.* In Tamhane and Bechhofer (1977) we proposed the minimax criterion (i.e. minimize $\max_{\mu\in\Omega} E_{\mu}(N)$) and showed that the maximum of $E_{\mu}(N \mid \mathscr{P}_2)$ over Ω occurs at EMC. A table of design constants for minimax $\mathscr{P}_2$, say $\mathscr{P}_2^*$, computed using the sharper bound (2.7) on the PCS are given in Tamhane and Bechhofer (1979). It would be of interest to compare $\mathscr{P}_L(a^*_{EMC})$ with $\mathscr{P}_2^*$ and explore the extent of gains achieved by going from two to higher number of stages. In making this comparison it must be kept in mind that in determining the design constants of $\mathscr{P}_L(a^*_{EMC})$ we placed certain restrictions on them for computational ease. Furthermore, we used the less sharp bound (2.8) on the PCS of $\mathscr{P}_L$. Thus the comparison is biased in favor of $\mathscr{P}_2^*$.

As a measure of the performance of $\mathscr{P}_L(\mathscr{P}_L(a^*_{EMC})$ or $\mathscr{P}_L(a^*_{LFC}))$ relative to that of $\mathscr{P}_2^*$ when both guarantee the same probability requirement (1.1) we define the relative efficiency (RE) of $\mathscr{P}_2^*$ with respect to $\mathscr{P}_L$ as follows :

$$\mathrm{RE}_{\mu}(\mathscr{P}_2^* : \mathscr{P}_L \mid k, \delta^*, P^*, \sigma, \gamma) = \frac{\gamma E_{\mu}(M \mid \mathscr{P}_L) + E_{\mu}(N \mid \mathscr{P}_L)}{\gamma E_{\mu}(M \mid \mathscr{P}_{22}^*) + E_{\mu}(N \mid \mathscr{P}_2^*)}. \quad \ldots \quad (5.3)$$

Again note that RE-values less than unity favor $\mathcal{P}_L$ over $\mathcal{P}_2^*$. The quantities $E_\mu(M \mid \mathcal{P}_2^*)$ and $E_\mu(N \mid \mathcal{P}_2^*)$ were computed using the table of design constants for $\mathcal{P}_2^*$ given in Tamhane and Bechhofer (1979) and the exact formulae for these quantities given in Tamhane (1978). The Monte Carlo estimates of $E_\mu(M \mid \mathcal{P}_L)$ and $E_\mu(N \mid \mathcal{P}_L)$ were obtained from Table 4. Based on these results the RE-values of $\mathcal{P}_2^*$ relative to $\mathcal{P}_L(a_{EMC}^*)$ were computed using (5.3) and they are given in Table 8.

By examining Table 8 we notice that at EMC, RE > 1 in many cases unless k is large; thus for $k = 10$, RE < 1 when $P^* = 0{\cdot}99$ and $\gamma = 0$ and for $k = 25$, RE < 1 in all cases except one. This is to be expected since $\mathcal{P}_2^*$ is based on a sharper lower bound on the PCS and no restrictions are imposed on its design constants; thus it is possible to achieve lower values of $E_{EMC}(N)$ using $\mathcal{P}_2^*$. However if k is large than $\mathcal{P}_L^*(a_{EMC}^*)$ (for $L > 2$) is able to achieve lower values of $E_{EMC}(N)$ in spite of the less sharp lower bound on the PCS used and the restrictions imposed on the design constants.

What is perhaps most striking is that RE < 1 in most cases at LFC and in all the cases at EDC achieving values less than $0{\cdot}5$ for $k = 25$ and $L = 5$ even when $\gamma > 0$, i.e., even after the cost of additional stages is taken into account. Thus $\mathcal{P}_L(a_{EMC}^*)$ provides significant improvement over $\mathcal{P}_2^*$ in large number of situations.

Acknowledgement. The author is grateful to the referee for making many useful suggestions for improvement in presentation and organization of the paper.

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Paper received : March, 1979.

Revised : March, 1980.